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Necessary Conditions for L_p Convergence of Lagrange Interpolation on an Arbitrary System of Nodes*

YING GUANG SHI

Computing Center, Chinese Academy of Sciences, P.O. Box 2719, Beijing, China 100080

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This paper gives powerful necessary conditions for convergence of Lagrange interpolation on an arbitrary system of nodes in $L_p(d\alpha)$ with $d\alpha$ belonging to the Szegő's class. This provides a partial answer to Problem XI of P. Turán [J. Approx. Theory **29** (1980), 33–34]. It is shown that in this case the asymptotics of distribution of the nodes must behave like the power asymptotics. © 1996 Academic Press, Inc.

1. INTRODUCTION AND MAIN RESULTS

This paper deals with mean convergence of Lagrange interpolation on an arbitrary system of nodes.

Denote by X a triangular matrix of nodes

$$1 \geqslant x_{1n} > x_{2n} > \cdots > x_{nn} \geqslant -1, \qquad n = 1, 2, \ldots . \tag{1.1}$$

The Lagrange interpolating polynomial of $f \in C[-1, 1]$ on X is defined by

$$L_n(X, f) := L_n(X, f, x) := \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x), \qquad n = 1, 2, \ldots, \tag{1.2}$$

where

$$l_{kn}(x) := \frac{\omega_n(x)}{(x - x_{kn})\, \omega_n'(x_{kn})}, \qquad k = 1, 2, \ldots, n, \quad n = 1, 2, \ldots \tag{1.3}$$

with $\omega_n(x) := \omega_n(X, x) := (x - x_{1n})(x - x_{2n}) \cdots (x - x_{nn})$, $n = 1, 2, \ldots$. For simplicity sometimes we also write x_k instead of x_{kn}, etc.

Dealing with mean convergence of $L_n(X, f)$ Turán proposed [10, pp. 33–34]

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Problem XI. Given $p>1$, what is a necessary and sufficient condition that

$$\lim_{n\to\infty}\int_{-1}^{1} |f(x)-L_n(X,f,x)|^p\,dx=0 \tag{1.4}$$

for every $f\in C[-1,1]$?

Let $d\alpha(x)$ be a finite Borel positive measure on the interval $I:=[-1,1]$, whose support is an infinite set. Let

$$P_n(d\alpha,x)=\gamma_n x^n+\cdots$$

($\gamma_n:=\gamma_n(d\alpha)>0$), be the orthonormal polynomials with respect to $d\alpha$. It is natural to propose a more general

Problem. Let $d\alpha$ be a measure supported in $[-1,1]$. Given $p>0$, what is a necessary and sufficient condition that

$$\lim_{n\to\infty}\int_{-1}^{1} |f(x)-L_n(X,f,x)|^p\,d\alpha(x)=0 \tag{1.5}$$

for every $f\in C[-1,1]$?

This is a difficult problem, only some necessary conditions for weighted mean convergence of Lagrange interpolation on an arbitrary system of nodes are obtained (see [6–8; 10, pp. 33–34] and the references therein). In particular, in [8] it is shown that, if (1.5) holds for all $f\in C[-1,1]$ for a regular measure $d\alpha$, which means $\lim_{n\to\infty}\gamma_n(d\alpha)^{1/n}=2$, then the asymptotic behaviour of $\omega_n(X,x)$ behaves like the regular (nth root) asymptotic behaviour of $\omega_n(d\alpha,x)=P_n(d\alpha,x)/\gamma_n(d\alpha)$. In this paper we intend to give powerful necessary conditions guaranteeing (1.5) for all $f\in C[-1,1]$ for $d\alpha\in S$ (the Szegő's class) [5, p. 246], which means

$$\int_{-1}^{1}\frac{\ln\alpha'(x)\,dx}{\sqrt{1-x^2}}>-\infty. \tag{1.6}$$

To state our results we need some definitions and notations. Write for $f\in C[-1,1]$ and $0<p<\infty$

$$\|f\|_{d\alpha,p}:=\left\{\int_I |f(x)|^p\,d\alpha(x)\right\}^{1/p},$$

$$\|f\|:=\max_{x\in I}|f(x)|,$$

$$\|L_n(X)\|_{d\alpha,p}:=\sup_{\|f\|\leqslant 1}\|L_n(X,f)\|_{d\alpha,p},$$

$$S_n(X, x) := \sum_{k=1}^{n} |(x - x_{kn})\, l_{kn}(x)|,$$

$$\gamma_n(X) := \sum_{k=1}^{n} \frac{1}{|\omega_n'(x_{kn})|},$$

$$\phi(z) := z + \sqrt{z^2 - 1}, \qquad z \in \mathbf{C},$$

where $\mathbf{C}$ is the complex plane and the branch of the square root is taken so that $\sqrt{z^2 - 1}$ behaves like z near infinity.

The main result of this paper is the following

THEOREM. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and $d\alpha \in S$. Let $0 < p < \infty$. If* (1.5) *holds for every $f \in C[-1, 1]$, then*

(a)
$$\frac{\gamma_n(X)}{2^n} \leqslant \text{const}; \tag{1.7}$$

(b)
$$\left|\frac{S_n(X, z)}{\phi(z)^n}\right| \leqslant \text{const}, \qquad z \in \Omega \tag{1.8}$$

holds for every compact set Ω in $\mathbf{C} \backslash I$;

(c)
$$0 < c_1 \leqslant \left|\frac{2^n \omega_n(X, z)}{\phi(z)^n}\right| \leqslant c_2, \qquad z \in \Omega \tag{1.9}$$

holds for every compact set Ω in $\mathbf{C} \backslash I$, where c_1 and c_2 are constants independent of n;

(d)
$$\left|\sum_{k=1}^{n} f(x_{kn}) - \frac{n}{\pi} \int_{-1}^{1} \frac{f(x)\, dx}{\sqrt{1 - x^2}}\right| \leqslant \text{const}\, \|f\|_\Delta, \tag{1.10}$$

whenever f is analytic in an open set $\Delta \supset I$, here

$$\|f\|_\Delta = \sup_{z \in \Delta} |f(z)|.$$

Remark 1. We point out that (b) $\Leftrightarrow$ (a) & (c) (Lemma 6) and (c) $\Leftrightarrow$ (d) (Lemma 4).

Remark 2. As it is well known, for a measure $d\alpha \in S$ the corresponding orthogonal polynomials $\omega_n(d\alpha, x) = P_n(d\alpha, x)/\gamma_n(d\alpha)$ must satisfy the conditions (a)–(d) [5, Theorem 3.5, p. 246; 1, 2]. In particular, Statement (d) is a representation of the asymptotic distribution of nodes [5, p. 249]. Thus our theorem shows that under the assumptions of the theorem the asymptotic behaviour of $\omega_n(X, x)$ behaves like the asymptotic behaviour of $\omega_n(d\alpha, x)$.

In the next section some auxiliary lemmas are given. In the last section we proceed the proof of the theorem.

2. Auxiliary Lemmas

Let $d\mu(\theta)$ be a finite Borel positive measure on the interval $[-\pi, \pi]$, whose support is an infinite set. Let [4, p. 204]

$$G(d\mu) := \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln \mu'(\theta)\, d\theta\right\}. \tag{2.1}$$

If $d\mu \in S$, that is

$$\int_{-\pi}^{\pi} \ln \mu'(\theta)\, d\theta > -\infty, \tag{2.2}$$

then the Szegő's function is defined by

$$D(z) := D(d\mu, z) := \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} \ln \mu'(\theta)\, d\theta\right\} \tag{2.3}$$

which is analytic in $|z|<1$ [5, p. 242]. For an arbitrary continuous function $F(\theta)$ of period 2π we have [9, p. 269]

$$\lim_{r\to 1-0} \int_{-\pi}^{\pi} F(\theta)\, |D(re^{i\theta})|^2\, d\theta = \int_{-\pi}^{\pi} F(\theta)\, \mu'(\theta)\, d\theta. \tag{2.4}$$

Put

$$\begin{aligned} d_n &:= d_n(d\mu) \\ &:= \min_{b_k} \frac{1}{2\pi}\int_{-\pi}^{\pi} |z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n|^p\, d\mu(\theta),\ z = e^{i\theta}. \end{aligned} \tag{2.5}$$

If $d\mu \in S$, then [9, p. 376]

$$\lim_{n\to\infty} d_n(d\mu) = G(d\mu). \tag{2.6}$$

Now we can state and prove the following crucial

Lemma 1. *Let $d\mu$ be a measure supported in $[-\pi, \pi]$ and $d\mu \in S$. Let $0 < p < \infty$. If $\rho_n(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ has no zeros in $|z| > 1$ and satisfies*

$$\int_{-\pi}^{\pi} |\rho_n(e^{i\theta})|^p\, d\mu(\theta) \leqslant \text{const}, \tag{2.7}$$

then

$$\left|\frac{\rho_n(z)}{z^n}\right| \leqslant \text{const}, \qquad |z| \geqslant R > 1. \tag{2.8}$$

Proof. Following the line of the proof of Theorem 12.1.1 in [9, p. 295] here we must consider the function

$$D(z)\,\psi_n^*(z)^{p/2} - 1 = [D(0)\,d_n^{-1/2} - 1] + d_{n1}z + d_{n2}z^2 + \cdots \tag{2.9}$$

where

$$\psi_n(z) = d_n^{-1/p}\rho_n(z), \qquad \psi_n^*(z) = z^n\bar{\psi}_n(z^{-1}). \tag{2.10}$$

The remainder of the proof is an almost word for word repetition of the proof of Theorem 12.1.1 in [9, p. 295], so we omit the details. ▮

As a consequence of Lemma 1 we state

LEMMA 2. *Let dα be a measure supported in* $[-1, 1]$ *and* $d\alpha \in S$. *Let* $0 < p < \infty$. *If*

$$\int_{-1}^{1} |2^n\omega_n(X, x)|^p \, d\alpha(x) \leqslant \text{const}, \tag{2.11}$$

then

$$\left|\frac{2^n\omega_n(X, z)}{\phi(z)^n}\right| \leqslant \text{const}, \qquad z \in \Omega, \tag{2.12}$$

whenever $\Omega \subset \mathbf{C}\backslash I$ *is compact.*

Proof. By introducing $Q_n(x) := 2^n\omega_n(X, x)$ and

$$x = \tfrac{1}{2}(z + z^{-1}) \tag{2.13}$$

we get

$$\begin{aligned} Q_n(x) &= \sum_{k=0}^{n} b_{n-k}x^k = \sum_{k=0}^{n} 2^{-k}b_{n-k}(z + z^{-1})^k \\ &= z^{-n}\sum_{k=0}^{2n} a_{2n-k}z^k := z^{-n}\rho_{2n}(z). \end{aligned} \tag{2.14}$$

It is easy to check that

$$a_k = a_{2n-k}, \qquad k = 0, 1, ..., n, \qquad a_0 = a_{2n} = 1 \tag{2.15}$$

and

$$|Q_n(x)| = |\rho_{2n}(z)|, \qquad z = e^{i\theta}. \tag{2.16}$$

Meanwhile we see that $\rho_{2n}(z) \neq 0$ for $|z| \neq 1$.

Now define the associated measure $d\mu(\theta)$ on $[-\pi, \pi]$ by

$$\mu(\theta) := \begin{cases} \alpha(\cos\theta) - \alpha(1), & \theta \in [-\pi, 0], \\ \alpha(1) - \alpha(\cos\theta), & \theta \in [0, \pi] \end{cases} \tag{2.17}$$

so that

$$d\mu(\theta) = |\sin\theta| \, d\alpha(\cos\theta) \tag{2.18}$$

and

$$\int_{-\pi}^{\pi} |\rho_{2n}(e^{i\theta})|^p \, d\mu(\theta) = \int_{-\pi}^{\pi} |Q_n(\cos\theta)|^p \, |\sin\theta| \, d\alpha(\cos\theta)$$

$$= 2 \int_{-1}^{1} |Q_n(x)|^p \, d\alpha(x). \tag{2.19}$$

Applying Lemma 1, this, together with (2.11), yields

$$\left| \frac{\rho_{2n}(z)}{z^{2n}} \right| \leqslant \text{const}, \qquad |z| \geqslant R > 1.$$

By virtue of (2.14) we have

$$\left| \frac{Q_n(x)}{z^n} \right| \leqslant \text{const}, \qquad |z| \geqslant R > 1,$$

from which (2.12) follows at once. ∎

LEMMA 3. *Let $d\alpha$ be a measure supported in $[-1, 1]$ and let $d\mu$ be defined by* (2.17). *Assume that $0 < p < \infty$ and*

$$e_n := e_n(d\alpha) := e_n(d\alpha, p)$$

$$:= \min_{b_k} \frac{1}{\pi} \int_{-1}^{1} |2^n x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n|^p \, d\alpha(x). \tag{2.20}$$

Then

$$e_n(d\alpha) \geqslant d_{2n}(d\mu) \geqslant G(d\mu). \tag{2.21}$$

Meanwhile

$$\lim_{n \to \infty} e_n(d\alpha) > 0 \tag{2.22}$$

if and only if $d\alpha \in S$.

Proof. By definition we have $d_n(d\mu) \geqslant d_{n+1}(d\mu)$, which by (2.6) implies $d_{2n}(d\mu) \geqslant G(d\mu)$. Meanwhile (2.19) yields $e_n(d\alpha) \geqslant d_{2n}(d\mu)$.

Let us prove the last conclusion. If $d\alpha \in S$ then (2.21) implies (2.22).

Conversely, let $p \geqslant 1$ and let $\phi_n(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a solution of (2.5). Since $\mu(-\theta) = \mu(\theta)$,

$$\overline{\phi_n(z^{-1})} = \bar{\phi}_n(z) = z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n$$

is also a solution of (2.5). Thus $\psi_n(z) = \frac{1}{2}[\phi_n(z) + \bar{\phi}_n(z)]$ is a solution of (2.5) and has real coefficients.

Let $z = e^{i\theta}$ and

$$\psi_n(z) = \sum_{k=0}^{n} b_{n-k} z^k = \sum_{k=0}^{n} b_{n-k}[\cos k\theta + i \sin k\theta], \qquad b_0 = 1.$$

If $T_k(x)$ denotes the kth Chebyshev polynomial of the first kind then

$$d_n(d\mu) = \frac{1}{2\pi} \int_{-1}^{\pi} |\psi_n(e^{i\theta})|^p \, d\mu(\theta) \geqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n} b_{n-k} \cos k\theta \right|^p d\mu(\theta)$$

$$= \frac{1}{\pi} \int_{-1}^{1} \left| \sum_{k=0}^{n} b_{n-k} T_k(x) \right|^p d\alpha(x)$$

$$= \frac{1}{\pi 2^p} \int_{-1}^{1} |2^n x^n + \cdots|^p \, d\alpha(x) \geqslant 2^{-p} e_n(d\alpha).$$

Thus, for $p \geqslant 1$, $d\alpha \in S$ follows at once from (2.6), (2.22), and the above inequality. Here we have used the fact: under the assumption (2.17), (1.6) is equivalent to (2.2).

If $p < 1$ then by Hölder's inequality we obtain $e_n(d\alpha, p) \leqslant \text{const}\; e_n(d\alpha, 1)$. This means $\lim_{n \to \infty} e_n(d\alpha, 1) > 0$ and hence $d\alpha \in S$. ▮

Developing and properly modifying the ideas in [1, 2] we can prove the following lemma, which is of independent interest.

LEMMA 4. *Let X be given in* (1.1). *Then Statement* (c) *is equivalent to Statement* (d).

Proof. ($\Rightarrow$) Let $\Delta \supset I$ be an open set. Choose a closed curve Γ in Δ surrounding I so that $\operatorname{dist}(\Gamma, I) > 0$ and put $\Omega = \Gamma$. Then, for an arbitrary sequence $\alpha(n)$ satisfying $\alpha(n) \to 0$ as $n \to \infty$, it follows from (1.9) that

$$\lim_{n \to \infty} \frac{\alpha(n)\, 2^n \omega_n(z)}{\phi(z)^n} = 0$$

holds uniformly on Ω. Thus by differentiation we conclude

$$\lim_{n \to \infty} \frac{\alpha(n)\, 2^n \omega_n(z)}{\phi(z)^n} \left\{ \frac{\omega_n'(z)}{\omega_n(z)} - \frac{n}{\sqrt{z^2-1}} \right\} = 0$$

holds uniformly on Ω. Since $\alpha(n)$ is arbitrary, by (1.9) we have

$$R_n(z) := \frac{\omega_n'(z)}{\omega_n(z)} - \frac{n}{\sqrt{z^2-1}} \tag{2.23}$$

satisfies $|R_n(z)| \leqslant \text{const}$, $z \in \Omega$. From (2.23) we obtain that for every f analytic in Δ

$$\frac{1}{2\pi i} \int_\Gamma f(z) \left\{ \frac{\omega_n'(z)}{\omega_n(z)} - \frac{n}{\sqrt{z^2-1}} \right\} dz = \frac{1}{2\pi i} \int_\Gamma f(z)\, R_n(z)\, dz.$$

Applying the residue theorem one has

$$\frac{1}{2\pi i} \int_\Gamma f(z) \frac{\omega_n'(z)}{\omega_n(z)}\, dz = \sum_{k=1}^{n} f(x_{kn}).$$

Meanwhile

$$\frac{n}{2\pi i} \int_\Gamma \frac{f(z)}{\sqrt{z^2-1}}\, dz = \frac{n}{\pi} \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx$$

and

$$\left| \frac{1}{2\pi i} \int_\Gamma f(z)\, R_n(z)\, dz \right| \leqslant \text{const}\, \|f\|_\Delta.$$

Then (1.10) follows at once.

($\Leftarrow$) Let $\Omega \subset \mathbf{C} \setminus I$ be an arbitrary compact set. Choose an open set Δ so that $\Delta \supset I$ and $\operatorname{dist}(\Delta, \Omega) > 0$, which is possible, since Ω is compact and $\Omega \cap I = \phi$. Then $f(x) = \ln(z - x)$ with $z \in \Omega$ is analytic in Δ. For this function (1.10) gives

$$\left| \ln \omega_n(z) - \frac{n}{\pi} \int_{-1}^{1} \frac{\ln(z-x)\, dx}{\sqrt{1-x^2}} \right| \leqslant \text{const}\, \|f\|_\Delta \leqslant c_0, \qquad z \in \Omega, \tag{2.24}$$

Now we need the following relation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\ln(z-x)\,dx}{\sqrt{1-x^2}}=\ln\left[\frac{1}{2}\,\phi(z)\right],\qquad z\in\mathbf{C}\setminus I, \tag{2.25}$$

which may be proved by the argument used in [3, Lemma 2.2]. It follows from (2.24) and (2.25) that

$$\left|\ln\frac{2^n\omega_n(X,z)}{\phi(z)^n}\right|\leqslant c_0,\qquad z\in\Omega,$$

which gives (1.9) provided we put $c_1=e^{-c_0}$ and $c_2=e^{c_0}$. ∎

For an estimation of the lower bound of $2^n\omega_n(z)/\phi(z)^n$ we have

Lemma 5. *Let X be given in* (1.1). *Then*

$$\left|\frac{S_n(X,z)}{\phi(z)^n}\right|\geqslant c>0,\qquad z\in\Omega \tag{2.26}$$

holds for every compact set Ω in $\mathbf{C}\setminus I$. Meanwhile, if (1.7) *holds, then*

$$\left|\frac{2^n\omega_n(X,z)}{\phi(z)^n}\right|\geqslant c_1>0,\qquad z\in\Omega \tag{2.27}$$

holds for every compact set Ω in $\mathbf{C}\setminus I$.

Here c and c_1 are constants independent of n.

Proof. Using the Lagrange interpolation formula based on the nodes $x_1, ..., x_n$, the $(n-1)$th Chebyshev polynomial on the first kind $T_{n-1}(z)$, $z\in\Omega$, can be expressed as

$$T_{n-1}(z)=\sum_{k=1}^{n}T_{n-1}(x_k)\,l_k(z)=\sum_{k=1}^{n}T_{n-1}(x_k)\,\frac{\omega_n(z)}{(z-x_k)\,\omega_n'(x_k)}.$$

Then

$$|T_{n-1}(z)|\leqslant|S_n(z)|\max_{1\leqslant k\leqslant n}\frac{1}{|z-x_k|}\leqslant|S_n(z)|\,d(z), \tag{2.28}$$

where $d(z)=1/\mathrm{dist}(z,I)$. Taking logarithms on both sides in (2.28) yields

$$\ln|T_{n-1}(z)|\leqslant\ln S_n(z)+\ln d(z)\leqslant\ln|S_n(z)|+c_3,$$

or in another form,

$$\left|\frac{S_n(z)}{T_{n-1}(z)}\right|\geqslant c_4>0,\qquad z\in\Omega.$$

From the formula [5, p. 239]

$$\lim_{n\to\infty} \frac{T_n(z)}{\phi(z)^n} = \frac{1}{2}, \qquad z \in \mathbf{C} \setminus I, \tag{2.29}$$

(2.26) follows at once.

Meanwhile, by definition we have

$$S_n(X, z) = \gamma_n(X)\, \omega_n(X, z). \tag{2.30}$$

Thus (2.27) is an immediate consequence of (2.26) and (1.7). ∎

LEMMA 6. *Statement* (b) *holds if and only if Statement* (a) *and Statement* (c) *hold.*

Proof. By (2.30) Statements (a) and (c) imply Statement (b).

Conversely, by the same arguments as in Lemma 4 it follows from (1.8) and (2.26) that Statement (d) holds. By Lemma 4 Statement (c) is true. Furthermore, by (2.30) Statements (b) and (c) yield Statement (a). ∎

As the final step of preliminaries we state a basic lemma.

LEMMA 7 [7]. *Let $d\alpha$ be an arbitrary measure supported in $[-1, 1]$ and $0 < p_0 \leqslant p \leqslant \infty$. Then for arbitrary system X of nodes*

$$\|S_n(X)\|_{d\alpha, p} \leqslant c(p_0)\, \|L_n(X)\|_{d\alpha, p}, \qquad n \geqslant 1. \tag{2.31}$$

3. PROOF OF THE THEOREM

If (1.5) holds for every $f \in C[-1, 1]$ then by the Banach theorem $\|L_n(X)\|_{d\alpha, p} \leqslant$ const. So by (2.31)

$$\|S_n(X)\|_{d\alpha, p} \leqslant \text{const}, \tag{3.1}$$

or equivalently

$$\gamma_n(X)\, \|\omega_n(X)\|_{d\alpha, p} \leqslant \text{const}. \tag{3.2}$$

Since $d\alpha \in S$, by (2.21)

$$2^n\, \|\omega_n(X)\|_{d\alpha, p} \geqslant [\pi G(d\mu)]^{1/p} > 0 \tag{3.3}$$

which by (3.2) implies Statement (a).

On the other hand, it is well known (cf. [7, (2.20)]) that for an arbitrary system X of nodes $\gamma_n(X) \geqslant 2^{n-2}$ from which we conclude $2^n\, \|\omega_n(X)\|_{d\alpha, p} \leqslant$ const. Then applying Lemmas 2 and 5 yields Statement (c).

Statement (b) follows from Statements (a) and (c) by Lemma 6. Meanwhile, Statement (d) follows from Statement (c) by Lemma 4.

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